MATH6031 Lecture 5

V : super vector space, i.e. $\mathbb{Z}/2\mathbb{Z}$-graded

$$V = V_0 \oplus V_1$$

$$\begin{matrix} & L \\ M & N \end{matrix}\Bigg) \qquad str(X) := tr(K) - tr(N)$$

If X is invertible, then its Berezinian is given by

$$Ber(X) = \det(K - LN^{-1}M)\det(N)^{-1}$$

- We have symmetric and exterior algebras of V

$$S^{\cdot}(V) \qquad \Lambda^{\cdot}(V)$$

- <u>Thm</u> If $\mathfrak{g}$ is a finite-dim$^{\underline{nl}}$ Lie algebra, there is an isom of graded algebras

$$HH^{\cdot}(\Lambda\mathfrak{g}^*, d_c) \xrightarrow{\sim} HH^{\cdot}(U(\mathfrak{g}))$$

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§ Duflo-Kontsevich isom. for Q-spaces

V : superspace

- $\mathcal{O}_V = S^{\cdot}(V^*)$: graded, supercommutative algebra of functions on V.

- $\mathfrak{X}_V := Der(\mathcal{O}_V) = S(V^*) \otimes V$: graded Lie super-algebra of vector fields on V

(cf. $V = \mathbb{C}^n$, $S(V^*) = \mathbb{C}[x_1, \dots, x_n]$

$TV = V \times \underset{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}}{V}$, $\Gamma(TV) = S(V^*)\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$)

$= S(V^*) \otimes V$

- $T_{poly}V := S^{\cdot}(V^* \oplus \Pi(V)) \cong \Lambda_{\mathcal{O}_V} \mathfrak{X}_V$: the $\mathfrak{X}_V$-module algebra of poly-vector fields on V

Gradings

- $\mathcal{O}_V$: internal grading - elts in V_i^* have deg $= i$
- $\mathfrak{X}_V$: restriction of grading on $\mathrm{End}(\mathcal{O}_V)$ - elts of V_i^* have deg i, and elts of V_i have deg $-i$.
- $T_{poly}V$: $\underline{\underline{3}}$ different gradings
  - one given by no. of arguments : elts of $\Lambda^k_{\mathcal{O}_V}\mathfrak{X}_V$ have deg k

    i.e. elts in V^* have deg $= 0$, and elts in V have deg $= 1$
  - one induced by $\mathfrak{X}_V$: elts in V_i^* have deg i and elts in V_i have deg $-i$; denoted by $|\cdot|$.
  - the total (or internal) degree : sum of previous two; elts of V_i^* have deg $0+i=i$, and elts of V_i have deg $1+(-i)=1-i$; denoted by $\|\cdot\|$.

We also have

- the $\mathfrak{X}_V$-module algebra D_V of diff. operators on V, which is the subalgebra of $\mathrm{End}(\mathcal{O}_V)$ generated by $\mathcal{O}_V$ and $\mathfrak{X}_V$.
- the $\mathfrak{X}_V$-module algebra $D_{poly}V$ of poly-diff operators on V, consisting of multilinear maps

$$\underbrace{\mathcal{O}_V \otimes \cdots \otimes \mathcal{O}_V} \to \mathcal{O}_V$$

which are differential operators in each argument.

Gradings

- D_V : restriction of grading on $\mathrm{End}(\mathcal{O}_V)$
- 3 different gradings on $D_{poly}V$:
  - one given by no. of arguments.
  - one induced by D_V ; denoted as $|\cdot|$.
  - one (the total or internal degree) given by sum of the above two ; denoted as $\|\cdot\|$.

Observation: $D_{poly}V$ is a subcomplex of the Hochschild complex of the graded, supercomm. algebra $\mathcal{O}_V$.

Prop The natural inclusion

$$I_{HKR} : (T_{poly}V, 0) \hookrightarrow (D_{poly}V, d_H)$$

is a quasi-isom. of complexes which induces an isom of algebras in cohomology.

Def A cohomological vector field on V is a deg 1 vector field $Q \in \mathfrak{X}_V$ which is integrable, i.e. $[Q, Q] = 2Q \cdot Q = 0$.

A Q-space is a superspace V equipped with a cohomological vector field Q.

Now we consider a Q-space (V, Q).

The adjoint action of Q on $T_{poly}V$ and $D_{poly}V$ is given by graded commutators (i.e. $[Q, \cdot]$)

$\rightsquigarrow$ we have DGAs

$$(T_{poly}V,\ Q\cdot) \quad \text{and} \quad (D_{poly}V,\ d_H + Q\cdot)$$

and also the HKR map

$$I_{HKR} : (T_{poly}V,\ Q\cdot) \longrightarrow (D_{poly}V,\ d_H + Q\cdot)$$

A spectral sequence argument $\Rightarrow$ I_{HKR} is a quasi-isom of complexes but DOESN'T preserve the products on cohomology.

The graded algebra of diff. forms on V is given by

$$\Omega^{\cdot}(V) := S(V^* \oplus \Pi V^*)$$

equipped with the following structures:

- $\forall\, x \in V^*$, write dx for the corr. elt. in ΠV^*.

  The de Rham differential d on $\Omega^{\cdot}(V)$ is defined by setting $d(x) = dx$ and $d(dx) = 0$.

- Action of ι of differential forms on polyvect. fields by contraction:

  $x \in V^*$ acts by left multiplication

and $dx \in \Pi V^*$ acts by derivation, i.e.,

for $y \in V^*$ and $v \in \Pi V$, we have

$$\iota_{dx}(y) = 0 \quad \text{and} \quad \iota_{dx}(v) = \langle x, v \rangle$$

Then we can define $\Xi \in \Omega^1(V) \otimes \mathrm{End}(V[1])$

(a (super-)matrix valued 1-form)

by $$\Xi^j_i := d\left(\frac{\partial Q^j}{\partial x^i}\right) = \frac{\partial^2 Q^j}{\partial x^k \partial x^i} dx^k$$

where $\{x^1, \dots, x^n\}$ are coordinates on V assoc. to a linear basis of V.

Note: change of basis in V

$\rightsquigarrow$ conjugation of Ξ by a constant matrix

So the elt

$$j(\Xi) := \mathrm{Ber}\left(\frac{1 - e^{-\Xi}}{\Xi}\right) \in \Omega^\cdot(V)$$

is independent of the choice of coord. on V.

Thm ✡ $$I_{HKR} \circ \iota_{j(\Xi)^{1/2}} : (T_{poly} V, Q\cdot) \longrightarrow (D_{poly} V, d_H + Q\cdot)$$

is a quasi-isom. of complexes which induces an algebra isom on cohomology.

§ Pf of the (extended) Duflo isom

Let $\mathfrak{g}$ be a finite-dim$^{\underline{nl}}$ Lie algebra

Take $\quad V = \Pi\mathfrak{g}$

Then $\quad \mathcal{O}_V \cong \Lambda^\bullet \mathfrak{g}^*$

Note : • grading on $\mathcal{O}_V$ = grading on Chevalley-Eilenberg complex $C^\bullet(\mathfrak{g}, k) = \Lambda^\bullet\mathfrak{g}^*$

• $\quad Q \longleftrightarrow d_C$

If $\{x^i\}$ are the (odd) coord. on $V = \Pi\mathfrak{g}$ associated a basis $\{e_i\}$ of $\mathfrak{g}$, then we have an identification

$$\mathcal{O}_V \longrightarrow \Lambda^\bullet\mathfrak{g}^*$$
$$x^{i_1}\cdots x^{i_p} \longmapsto \varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_p}, \quad 1 \le i_1 < \cdots < i_p \le n$$

where $\{\varepsilon^i\}$ is the dual basis of $\{e_i\}$.

So Q can be written as

$$Q = -\frac{1}{2} c^i_{jk} x^j x^k \frac{\partial}{\partial x^i}$$

where c^i_{jk} are structure consts of $\mathfrak{g}$ w.r.t. $\{e_i\}$.

Q has deg 1 and total degree 2.

Lemma 1 We have an identification of DGAs

$$(T_{poly}V, Q\cdot) \xrightarrow{\sim} (C^\bullet(\mathfrak{g}, S(\mathfrak{g})), d_C)$$
$$x^{i_1}\cdots x^{i_p}\partial_{x^{j_1}}\wedge\cdots\wedge\partial_{x^{j_t}} \longmapsto \varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_p}\otimes e_{j_1}\cdots e_{j_t}$$

On the other hand, we have

Lemma 2 There is an identification of DGAs

Lemma 2 There is an identification of DGAs

$$(D_{poly}V,\ d_H + Q\cdot) \xrightarrow{\sim} \left(C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*),\ d_H + d_c\right)$$

Recall we have a quasi-isom of complexes

$$\left(C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*),\ d_H + d_c\right) \xrightarrow{\sim} \left(C^\cdot(\mathfrak{g}, U(\mathfrak{g})),\ d_c\right)$$

$\hookrightarrow$ a commutative diagram

$$\begin{tikzcd}
(T_{poly}V,\ Q\cdot) \arrow[r, "I_{HKR}\circ\iota_{j(\Xi)^{1/2}}"] \arrow[d, equal, "\text{Lemma 1}"'] \arrow[dr, phantom, "\circlearrowright"] & (D_{poly}V,\ d_H + Q\cdot) \overset{\text{Lemma 2}}{=} \left(C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*),\ d_H + d_c\right) \arrow[d, "\wr"] \\
(C^\cdot(\mathfrak{g}, S(\mathfrak{g})),\ d_c) \arrow[r, "I_{PBW}\circ J^{1/2}"] & (C^\cdot(\mathfrak{g}, U(\mathfrak{g})),\ d_c)
\end{tikzcd}$$

Under the identification $V[1] \cong \mathfrak{g}$,

the (super-)matrix valued 1-form Ξ is given by

$$\Xi = ad$$

reasoning: $\quad Q = -\frac{1}{2} c^i_{jk} x^j x^k \frac{\partial}{\partial x^i}$

$$\Rightarrow \quad \Xi^i_j = d\left(\frac{\partial}{\partial x^j} Q^i\right) = -c^i_{jk}\, dx^k = c^i_{kj}\, dx^k$$

The claim follows by evaluating on e_k #

Hence Thm ✡ $\Rightarrow$ extended Duflo isom.

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§ Strategy of pf of Thm ✡.

- The pf by a homotopy argument:

  we construct a quasi-isom. of complexes

$$\mathcal{U}_Q : (T_{poly}V, Q\cdot) \longrightarrow (D_{poly}V, d_H + Q\cdot)$$

and a degree -1 map

$$\mathcal{H}_Q : T_{poly}V \otimes T_{poly}V \longrightarrow D_{poly}V$$

satisfying the homotopy equation

$$\boxed{\begin{aligned} &\mathcal{U}_Q(\alpha) \cup \mathcal{U}_Q(\beta) - \mathcal{U}_Q(\alpha \wedge \beta) \\ &\overset{(*)}{=} (d_H + Q\cdot)(\mathcal{H}_Q(\alpha,\beta)) + \mathcal{H}_Q(Q\cdot\alpha,\beta) + (-1)^{\|\alpha\|}\mathcal{H}_Q(\alpha, Q\cdot\beta) \\ &\forall\ \alpha,\beta \in T_{poly}V \end{aligned}}$$

- For any $\alpha, \beta \in T_{poly}V$ and fcns $f_1, \dots, f_m$, we set

$$\mathcal{U}_Q(\alpha)(f_1,\dots,f_m) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}_{n+1,m}} W_\Gamma B_\Gamma(\underbrace{\alpha, Q, \dots, Q}_{n+1})(\underbrace{f_1,\dots,f_m}_{m})$$

and

$$\mathcal{H}_Q(\alpha,\beta)(f_1,\dots,f_m) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}_{n+2,m}} \widetilde{W}_\Gamma B_\Gamma(\underbrace{\alpha,\beta, Q, \dots, Q}_{n+2})(\underbrace{f_1,\dots,f_m}_{m})$$

Here: $\mathcal{G}_{n,m}$ is a set of suitable directed graphs with 2 types of vertices to which we associate scalar weights W_Γ and $\widetilde{W}_\Gamma$ and poly-diff operators B_Γ.

- Can prove that $\mathcal{U}_Q(\alpha\wedge\beta)$ and $\mathcal{U}_Q(\alpha) \wedge \mathcal{U}_Q(\beta)$ (resp. RHS of $(*)$) are given by graph sum formulas similar to that for $\mathcal{H}_Q$ but with <u>new</u> weights W^0_Γ and W^1_Γ

(resp. W_I^2)

So (*) reduces to showing that

$$W_I^0 = W_I^1 + W_I^2 .$$

<u>related directions</u>

- operadic approach (Tamarkin)
  - ~ action of the Grothendieck - Teichmuller group
- formality for Lie algebroid pairs (Liao - Stienon - Xu)

QFT approach